BOUNCING BALL EXPERIMENT

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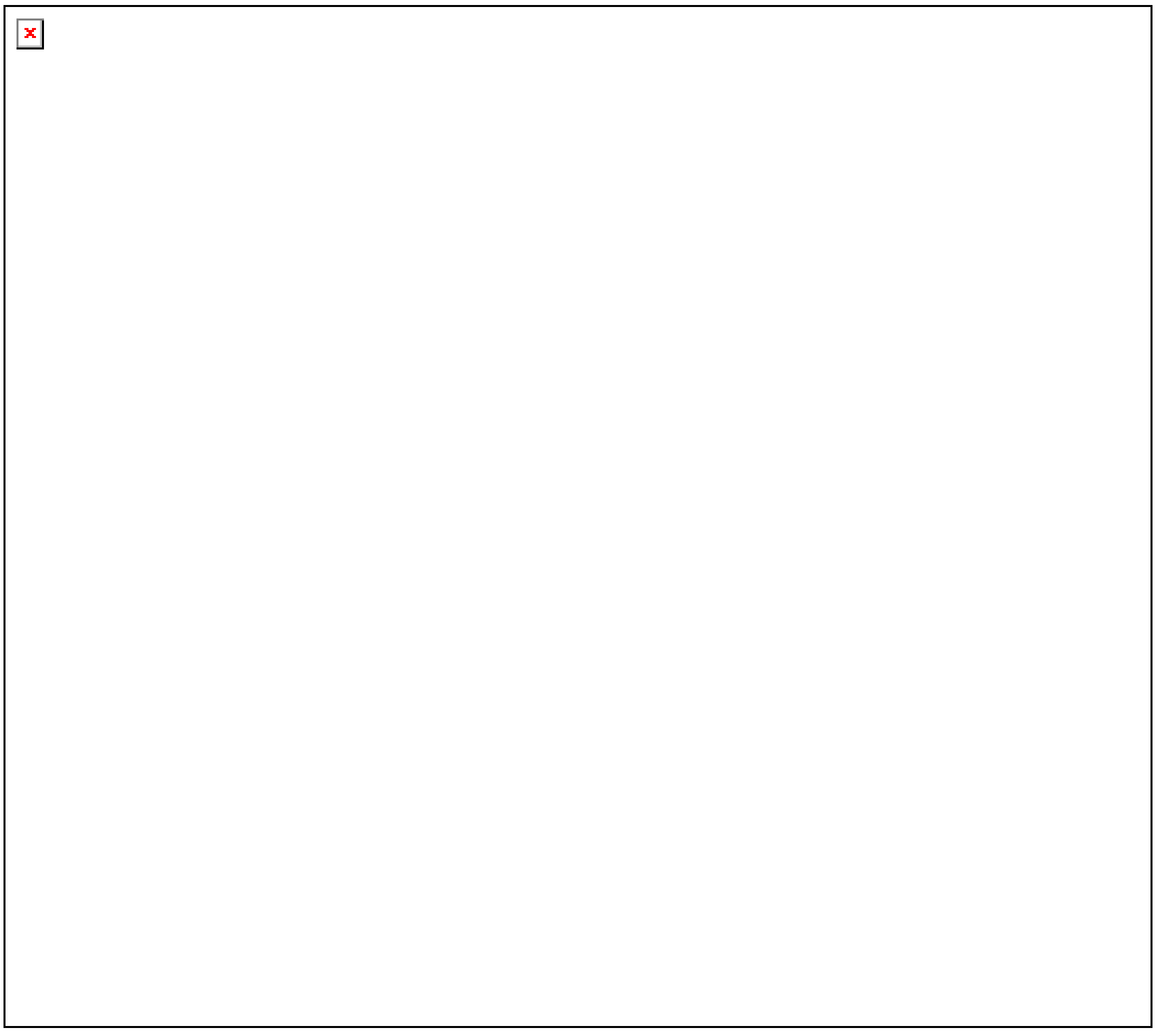
Aim

To investigate the behaviour of the bouncing ball system, especially the period doubling route to chaos, and various stable modes.

Apparatus

Speaker, Function Generator, Steel Ball bearings of three different sizes, Piezoelectric sensor, Cathode Ray Oscilloscope, Breadboard.

The apparatus was arranged as schematically shown below in the diagram:



Generally the piezoelectric is attached closely to the surface of the speaker so that the only signal the piezoelectric gives is during the impact of the ball. However, in our case, there was a gap between the piezoelectric and the speaker, which made the piezoelectric pick up a small sinusoidal signal from the vibrations of the speaker. On top of this signal there were spikes whenever the ball hit the piezoelectric. We found it to our advantage to retain this arrangement.

Theory

The height of the speaker surface varies as

-(1)

(the 1 is added to ensure that the amplitude always remains positive)

Initially due to the low coefficient of restitution α, for low amplitudes of oscillation, the ball will not bounce on the speaker but will move with the speaker surface. This continues until a critical value of $A=A_c = g/\omega^2$ when the acceleration of the speaker is greater than that of gravity. At this point the ball will start to bounce. When the ball bounces, it strikes the surface of the piezoelectric sensor and generates a brief sharp spike in the voltage.

Now, in the reference frame of the table:

-(2)

where v_k is the velocity after the k^{th} impact and v'_k is the velocity just before the k^{th} impact. The bar indicates that the velocities are in the reference frame of the table.

Taking into account the table velocity u, we have, in the reference frame of the laboratory:

-(3)

where -(4)

Also, between the k^{th} impact and the $k+1^{th}$, the velocity of the ball is given by

-(5)

Given that the k^{th} impact took place at a time t_k, at a time t , the position of the ball is given by

-(6)

and the position of the table by

The condition for impact at a time t_{k+1}, is that the difference between these two heights should be zero at $t=t_{k+1}$. This gives:

-(7)

Substituting , we get

-(8)

We also get, on substituting equations (4) and (5) in (3):

-(9)

Equations (8) and (9) are the exact equations or map of the bouncing ball problem. However the equations are implicit in θ_{k+1}; that is it cannot be extracted and expressed in terms of other quantities. This makes the map tougher to solve.

High bounce approximation:

In this approximation, we consider the variation in the height of the speaker surface to be negligible in comparison to the bouncing of the ball. However, the speaker continues to impart on impact a momentum that varies sinusoidally with time.

Thus, we have:

-(a)

that is the velocity just before the $k+1^{th}$ impact is equal in magnitude and opposite in direction to the velocity just after the k^{th} impact.

We also have, from (5):

-(b)

From (a) and (b) we get

-(c)

From (3) and (a) we have

-(d)

Using (4) and (c) in (d),

-(e)

Using the non dimensional variables

we get the simplified map:

-(f)

-(g)

When $\alpha = 1$, this map is known as the standard map. It consists of two explicit equations, which are easier to handle than equations (8) and (9).

The high bounce approximation shares many of the same qualitative properties of the exact model for the bouncing ball system, and it serves as the starting point for several analytic calculations. However, the high bounce model fails in at least two major ways to model the actual physical system . First, the high bounce model can generate solutions that cannot possibly occur in the real system. These unphysical solutions occur for very small bounces at negative table velocities, where it is possible for the ball to be projected downward beneath the table. That is, the ball can pass through the table in this approximation. Second, this approximation cannot reproduce a large class of real solutions, called ``sticking solutions'', in which though the table's maximum acceleration is much greater than *g*, the ball can become stuck. An infinite number of impacts can occur in a finite stopping time. The sum of the times between impacts converges in a finite time much less than the table's period, *T*. The ball gets stuck again at the end of this sequence of impacts and moves with the table until it reaches the phase $\theta = \sin^{-1}(g/A\omega^2)$. This type of sticking solution is an eventually periodic orbit. After its first time of getting stuck, it will exactly repeat this pattern of getting stuck, and then released, forever. Fundamentally, these solutions are not given by the map because the map in the high bounce approximation is invertible, whereas the exact model is not invertible. In the exact model there exist some solutions--in particular the sticking solutions--for which two or more orbits are mapped to the same identical point. Thus the map at this point does not have a unique inverse.

Procedure

The speaker was fed a sine wave from the function generator, and the frequency was kept fixed, and the amplitude of vibration varied. The steel ball bearing was placed on the surface of the piezoelectric, which had been taped to the surface of the speaker. The output of the piezoelectric was observed on the CRO display.

Observations

Schematics of motion of speaker surface and ball in periods 1, 2, T' (when ball makes one bounce for two vibrations of the speaker), and when there is no ball:

No ball:

Period 1:

The ball bounces to the same height each time. The period is the same as that of the speaker.

Period 2:

The ball bounces to height h1 and then h2 and then once again h1. Note also that the ball no longer lands at the same phase of the speaker. There has been a bifurcation in the heights and the phases.

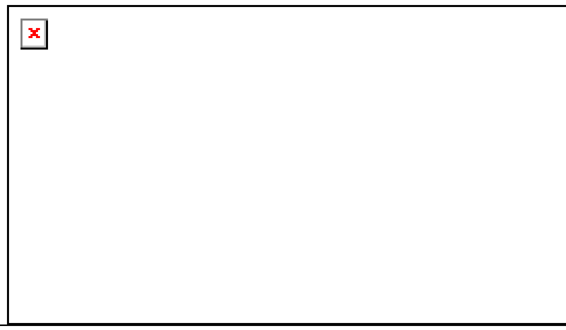
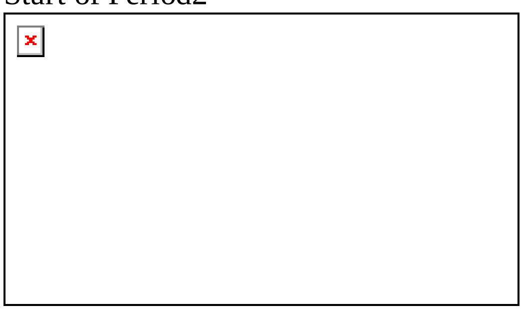
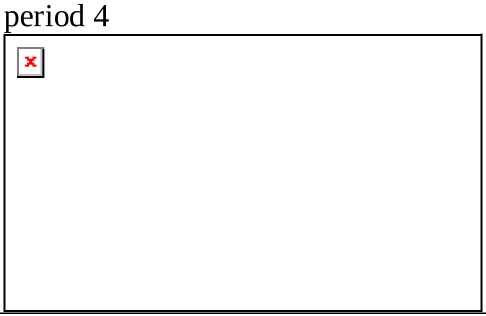
Period T'

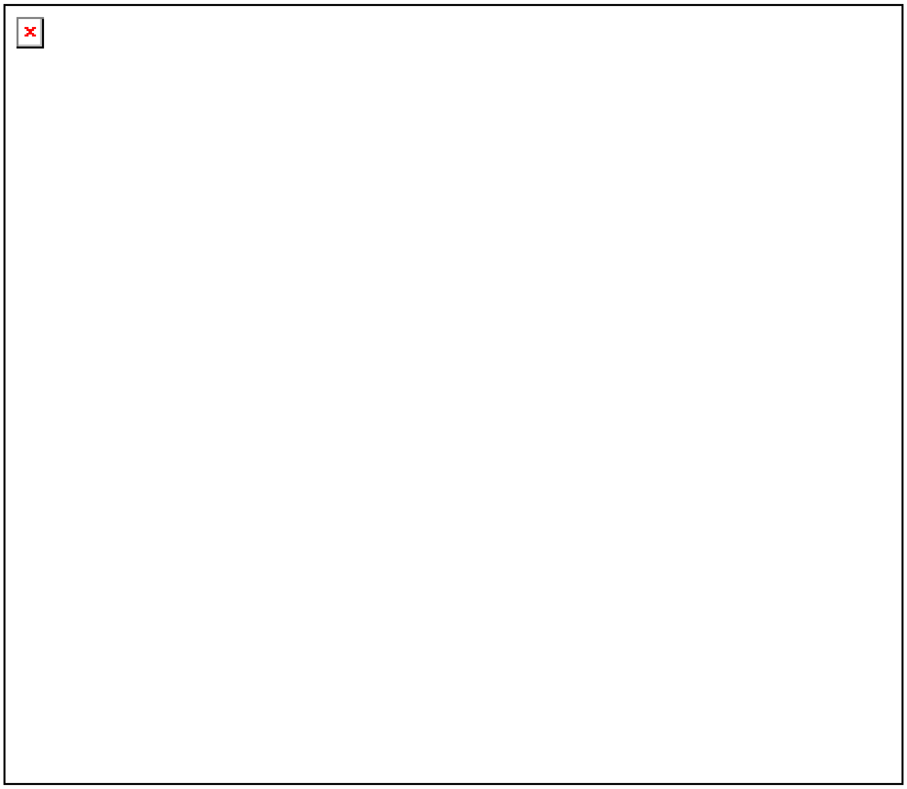
This mode is quite stable, and is followed (on further increase in the amplitude) by another bifurcation into T' period 2, and after that chaos.

<u>Data:</u>

<u>Amplitude Response:</u>

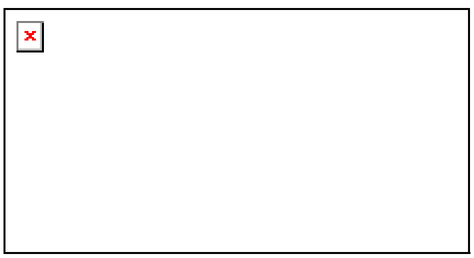
1) For medium size ball, at 73 Hz:

Parameter (Volts)	Behaviour	Height of spikes (V)
0.4	Period 1	0.8
0.5	Period 1 	1.75
0.6	Period 1	2.2
0.64	Period 1	2.6
0.76	Start of Period2 	3.3, 3
0.80	Period 2	3.4, 2.8
0.82	Period 2	3.5, 2.7
0.84	Period 2	3.7, 2.6
0.88	period 4 	3.7, 3.5, 2.8, 2.6

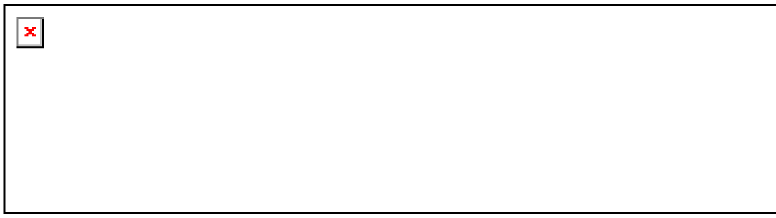


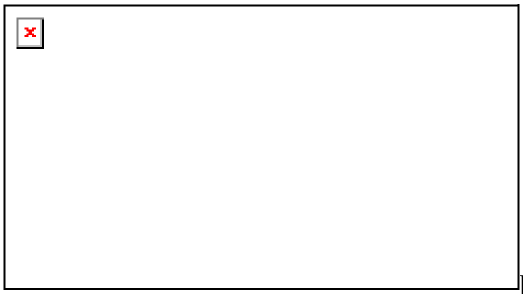
2) Amplitude response for medium ball at 73 Hz:

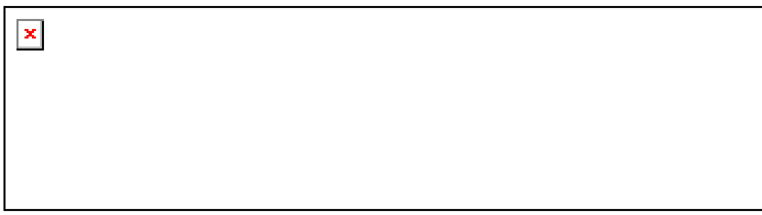
Parameter (volts)	Behaviour of system	Time intervals (ms)
0.72	Start of period 2	15,13
0.92	Start of period 4	
1.02	Stable period 4	16.5,12
1.12	Start of T’	29
1.34	Start of T’ period 2	30,28
1.95	Start of T’’	43
2.75	Start of T’’’	57



period T’


T’ period 2


period T’’


period T’’’

3) Amplitude response for the largest ball at 73Hz

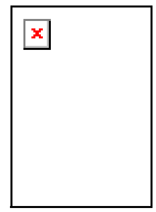
Parameter amplitude (volts)	Behaviour
0.34	Period 1
0.52	Period 1
0.76	Start of Period 2
0.8	Period 2
0.82	Period 2
0.88	Start of period 4
0.96	Period 4
1.08	Start of T’
1.26	Start of T’ period 2
2.2	Start of T’’

<u>**Using the XY mode to observe behaviour**</u>

The time picture of the piezoelectric output is not use to observe the behaviour of the system, as the system is constantly changing slightly. The heights of the spikes fluctuate constantly, even when in stable modes. With these fluctuations, it is not easy to judge when bifurcation takes place, especially as the difference in heights is initially very small.

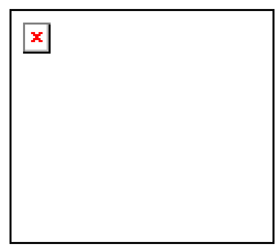
The answer to this is also the reason why we retained the base sinusoidal signal in the piezoelectric output. Inputting the piezoelectric output on Channel 1 of the CRO and the function generator signal on Channel 2, we can look at the outputs in XY mode.

Initially when there is no bouncing, the XY mode displays an ellipse. This is because both signals are sinusoidal, albeit out of phase slightly. The phase difference may be because a delay in the response of the piezoelectric to the speaker vibration, or due to the inductance of the electromagnet coil in speaker, which creates a phase difference with the function generator's oscillating current.



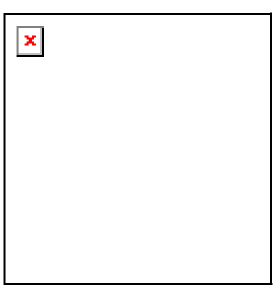
<u>No bouncing</u>

When the ball starts to bounce, the spike in the time mode signal appears as a spike in the ellipse. As the ball strikes the speaker at the same phase each time, the spike stays in one position.



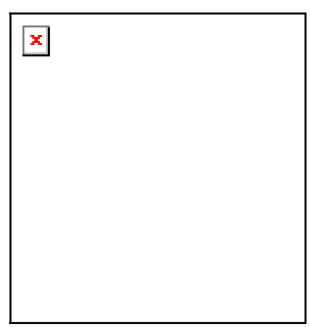
<u>Period 1</u>

When the bifurcation takes place, there is a bifurcation in height as well as *phase*. It is this phase bifurcation hat shows up prominently in the XY mode. The ball is now hitting the speaker at two different phases. The one spike has split into two. Initially they are close together, but as the parameter value is increased, the distance between the two spikes increases.



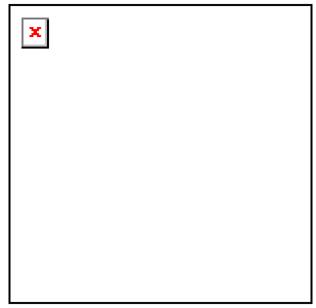
Increasing parameter ->

Period 2



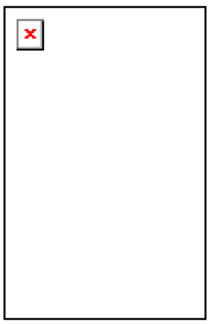
When the second bifurcation takes place, each of the two spikes splits into two.

Period 4

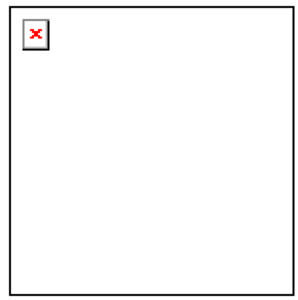


Beyond this chaos is observed, where the spikes in both the time mode and XY mode jump about randomly.

Then the T' mode appears, first as a bump, which then as the parameter is increased, develops into a spike.



Period T'



The XY mode is easy to observe, and soon became our most reliable method of determining when bifurcation had taken place. On determining bifurcation, we would switch back to time mode and take the values.

Frequency response

With increasing frequency , up to some frequency, the system exhibits bifurcation and the period doubling route to chaos. After that the behaviour is confused. For the bifurcation diagram frequency is the independent variable and the time intervals between the spikes the dependent variable undergoing bifurcation.

1) Frequency response of the system (for medium ball, at constant amplitude 1.5V)

Value of parameter (Hz)	Behaviour of system
55	Period 1
60	Start of Period 2
61	Start of Period 4
62	Start of period T’
67	Start of period T’ period 2 (bifurcation)
72	Start of new period T’
82	Start of shifted period T’
~140	Period 5

In the shifted period T’, the spikes (for the impact of the ball) no longer occurred at the peak of the sinusoidal signal of the piezoelectric, but shifted down from the peak.

Useful data was nearly impossible to attain as consecutive changes in behaviour would often take place within 1 Hz, which was the least count of the function generator.

Beyond 82 Hz, the behaviour became increasing confused, with period 4 seemingly reappearing, and at one point what appeared to be a period 2.

At approximately 140Hz, a somewhat stable period 5 occurred.

Behaviour of balls of different sizes at a frequency of 75 Hz.

Behaviour	Value of parameter for Small ball (Volts)	Value of parameter of Medium ball (Volts)	Value of parameter for Big ball (Volts)
Start of period 1	0.4	0.35	0.34
Start of period 2	0.84	0.9	0.9
Start of period 4	1.02	1.0	0.98
Start of period T'	1.46	1.4	1.32
Start of period T''	2.3	2.2	

From this data it can be seen that the values at which the system changes behaviour appear to be (within experimental error) independent of the size of the ball. This fits with the theory which was independent of the size and mass of the ball, depending only on the coefficient of restitution, which in this case is approximately the same for all three balls, as the contact is between steel and the piezoelectric surface.

ERRORS:

Least count of function generator: 1 Hz
Least count used in CRO display (voltage): 0.01V
Least count used in CRO display (time): 1ms

Precautions must be taken against parallax error. Also, the hysteresis of the system should be taken into account. Readings should be taken when the ball is moving as little as possible horizontally.

RESULTS:

- Due to non observation of period 8 bifurcation, the value of Feigenbaum delta could not be calculated.
- Due to wildly oscillating peak heights, the best estimate of Feigenbaum alpha that could be made was 1.6, using the extremal heights.
- The values of the parameter at which bifurcation takes place are independent of ball size.
- The values of the parameter at which bifurcation takes place are dependent on frequency.
- During bifurcation there is a split in the heights to which the ball bounces, and also in the phase at which they strike the table.
- Periods T', T'' and T''' are quite stable, the first being *very* stable.

Deficiencies and Required Improvements

No account taken of hysteresis: Being nonlinear, the system has a memory of initial conditions and displays hysteresis. Therefore, the values of the parameter at which bifurcation occurs varies according to whether we approach the value while decreasing the parameter or increasing it. Initial sets of data were not taken keeping this in mind.

Lack of fine tuning in amplitude and frequency: The function generator did not offer enough sensitivity in the variation of amplitude or frequency. Also, due to the impedances of the various circuit elements, only around 3V could be obtained across the speaker. To obtain the full 20 V, a unity gain buffer is required. A potential divider would then enable sufficient fine tuning of amplitude, and a LCR circuit may be used to adjust the frequency.

Better confinement required: To properly study the system the ball should move as closely as possible in one dimension. Due to imperfections and distortions on the ball and piezoelectric surface, the ball moves along the surface of the piezoelectric, changing the signal received and varying the heights of the spikes. Better confinement can be achieved by slightly curving the piezoelectric into a concave surface. This was attempted by attaching the piezoelectric to a watchglass, but the weight of the glass reduced the amplitude of vibration.

Vibration free table: Ideally the set-up would be mounted on a vibration free table so as to reduce outside influences on the system.

Bigger piezoelectric: Beyond a certain amplitude it is impossible to continue readings as the ball wanders off the surface of the piezoelectric. While this problem would partly be addressed by better confinement, a bigger piezoelectric would allow readings to be taken for a longer range.

Spectral analyzer: This would enable the detection of new Fourier frequencies when bifurcation occurred.

Data logger: This would allow easy interfacing with a computer and storage of data.

Phase diagram circuit: In a paper in Am. J. Phys 55(4), April 1987, Mello and Tufillaro describe a circuit which would allow direct plotting of the phase space of the bouncing ball on the CRO display.

Balls of different materials: Though system behaviour does not change with the size of the ball, it does change with the coefficient of restitution. In the present experiment we used only steel ball bearings. Plastic beads were used, but these gave an insufficient signal on striking the piezoelectric. Balls of heavy but different materials are required to note the variation in behaviour with alpha.

Note:

The piezoelectric was not calibrated, so the heights to which the ball bounced are not known, though for the initial periods they are less than a millimeter.

We were unable to estimate the coefficient of restitution: however, we determined it to be very low, bouncing not more than twice (by the evidence of the CRO display) with the second bounce being hardly visible. Thus α is very close to zero.

In a paper in Am. J. Phys., Vol. 56, no. 12, December 1988, Zimmerman and Celaschi comment that "in the limit α tending to zero (high damping) the ball contacts the table at phases that are separated by nearly $2\pi m$, m=1,2,3,…

In this case, there exists a relation between the phases ϕ_{N+1} and ϕ_N which denote the phases at which the ball strikes the table at the N+1th and Nth impact respectively. The relation is:

This can be thought of as a continuous function

which has a maximum at .

This describes the behaviour of the system for phases greater than ϕ_c. Phases smaller than ϕ_c start to appear in the iteration sequence near the onset of period four bifurcation. The lack of recursion for phases smaller than ϕ_c causes, in the high damping limit, the truncation of the universal route to chaos (by period doubling)."

Taking into account the low coefficient of restitution in the given experimental system, it is possible that this may explain why periods eight and beyond were not easily observed.

References:

1. Tufillaro and Albano, Am. J. Phys. 54 (10) October 1986
2. Mello and Tufillaro, Am. J. Phys. 55 (4) April 1987
3. Zimmerman and Celaschi, Am. J. Phys., Vol. 56, no. 12, December 1988
4. An experimental approach to nonlinear dynamics and chaos, by Nicholas B. Tufillaro, Jeremiah Reilly, and Tyler Abbott (web version, available at http://www.drchaos.net/drchaos/Book/node1.html)